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Integrable discretizations for Toda-type lattice soliton equations

Zuonong Zhu†‡ and Hongci Huang‡

† CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
‡ Department of Mathematics, Hong Kong Baptist University, Hong Kong, People's Republic of China

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Abstract. From a proper 2×2 discrete isospectral problem, a new integrable lattice soliton system is proposed. Integrable discretizations of a general Toda-type lattice soliton equation associated with the discrete isospectral problem are established. The Lagrangian and Newtonian forms of integrable discretizations of Toda-type lattice equations which occur in the literature are given uniformly and some new integrable discretizations of the Toda-type lattice are obtained.

1. Introduction

The study of the lattice soliton equations has received considerable attention in recent years. Many lattice soliton equations have been proposed, such as the Ablowitz–Ladik lattice [1–3], the Toda lattice [4, 5], the Lotka–Volterra lattice [6, 7], the Bogoyavlensky lattice [8, 9], the Belov–Chaltikian lattice [10], the differential-difference KdV equation [11], the Suris lattices [12–14] and so on. Given a discrete spectral problem and its continuous-time evolution problem

$$E\psi_n = U_n(u,\lambda)\psi_n \qquad \frac{\mathrm{d}\psi_n}{\mathrm{d}t} = N_n(u,\lambda)\psi_n \tag{1.1}$$

where U_n and N_n are two proper matrices, E is a shift operator defined by

$$Ef_n := f_{n+1} \qquad n \in \mathbb{Z}.$$

$\psi_n := \psi(n,t,\lambda)$, $u := u(n,t) = (u_1(n,t),\ldots,u_s(n,t))^T$ is a potential function, and λ is a spectral parameter, the compatibility condition of (1.1) is $E\mathrm{d}\psi_n/\mathrm{d}t = \mathrm{d}E\psi_n/\mathrm{d}t$, i.e. the following discrete zero curvature equation:

$$U_{n,t} + \frac{\partial U_n}{\partial\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t} - (EN_n)U_n + U_nN_n = 0 \tag{1.2}$$

where $\mathrm{d}\lambda/\mathrm{d}t = a\lambda$ with $a = 0$ for isospectral problem or $a = 1$ for the nonisospectral problem. The lattice equation derived from equation (1.2) is integrable in the Lax sense. Recently, we [15] considered a discrete spectral problem

$$\psi_{n+1} = U_n(u,\lambda)\psi_n \qquad U_n(u,\lambda) = \begin{pmatrix} \lambda p_n - \lambda^{-1} & q_n \\ r_n & \lambda s_n \end{pmatrix}. \tag{1.3}$$

By setting the continuous-time evolution equation

$$\frac{\mathrm{d}\psi_n}{\mathrm{d}t} = N_n\psi_n \qquad N_n = \begin{pmatrix} a_n + \mu\lambda^{-2} & \lambda^{-1}b_n \\ \lambda^{-1}c_n & d_n \end{pmatrix} \tag{1.4}$$

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where

$$a_n = -\mu q_n r_{n-1} - an + b \qquad b_n = -\mu q_n \qquad c_n = -\mu r_{n-1} \qquad d_n = f(p_n, q_n, r_n, s_n)$$

with μ and b arbitrary constants and f an arbitrary function, new integrable lattice systems were given. For the isospectral problem (1.3), suppose $s_n = \delta$, $q_n r_n = \beta p_n - \delta$ with β and δ arbitrary constants, and set $d_n = 0$, we obtain a lattice system from equation (1.2),

$$\begin{aligned} \dot{p}_n &= -\mu p_n \left(\frac{q_{n+1}(\beta p_n - \delta)}{q_n} - \frac{q_n(\beta p_{n-1} - \delta)}{q_{n-1}} \right) \\ \dot{q}_n &= q_n(\mu p_n + b) - \beta\mu q_{n+1} p_n. \end{aligned} \tag{1.5}$$

Set $p_n \to \mathrm{e}^{\epsilon p_n}$, $q_n \to \mathrm{e}^{q_n}$, with ϵ an arbitrary constant, equation (1.5) could be written in the form

$$\begin{aligned} \dot{p}_n &= \frac{\mu}{\epsilon}(\delta D \mathrm{e}^{q_n - q_{n-1}} - \beta D \mathrm{e}^{q_n - q_{n-1} + \epsilon p_{n-1}}) \\ \dot{q}_n &= \mu \mathrm{e}^{\epsilon p_n}(1 - \beta \mathrm{e}^{q_{n+1} - q_n}) + b \end{aligned} \tag{1.6}$$

which is a general Toda-type lattice soliton equation, where D is the difference operator defined by $Df_n = f_{n+1} - f_n$. Many famous lattice equations can be derived from equation (1.6) with the proper choice of parameters, as shown in table 1.

Table 1. Special Toda-type lattice soliton equations.

Parameters		Lattice equations in the Newtonian form
$\epsilon = \mu = \delta = 1, \quad b = \beta = 0$	(1)	$\ddot{q}_n = \dot{q}_n(\mathrm{e}^{q_{n+1} - q_n} - \mathrm{e}^{q_n - q_{n-1}})$
$\epsilon = \mu = 1, \quad \beta = -g^2, \quad b = \delta = 0$	(2)	$\ddot{q}_n = \dot{q}_{n+1}\dot{q}_n \frac{g^2 \mathrm{e}^{q_{n+1} - q_n}}{1 + g^2 \mathrm{e}^{q_{n+1} - q_n}} - \dot{q}_n \dot{q}_{n-1} \frac{g^2 \mathrm{e}^{q_n - q_{n-1}}}{1 + g^2 \mathrm{e}^{q_n - q_{n-1}}}$
$\epsilon = \mu = 1, \quad \beta = -g^2, \quad \delta \to \delta g^2, \quad b = 0$	(3)	$\ddot{q}_n = \dot{q}_n\dot{q}_{n+1} \frac{g^2 \mathrm{e}^{q_{n+1} - q_n}}{1 + g^2 \mathrm{e}^{q_{n+1} - q_n}} - \dot{q}_{n-1} \dot{q}_n \frac{g^2 \mathrm{e}^{q_n - q_{n-1}}}{1 + g^2 \mathrm{e}^{q_n - q_{n-1}}} + \delta g^2 \dot{q}_n (\mathrm{e}^{q_{n+1} - q_n} - \mathrm{e}^{q_n - q_{n-1}})$
$\mu = -b = \epsilon^{-1}, \quad \delta = \epsilon^2, \quad \beta = 0$	(4)	$\ddot{q}_n = (1 + \epsilon \dot{q}_n)(\mathrm{e}^{q_{n+1} - q_n} - \mathrm{e}^{q_n - q_{n-1}})$
$\mu = -b = \epsilon^{-1}, \quad \delta = 0, \quad \beta = -\epsilon^2$	(5)	$\ddot{q}_n = (1 + \epsilon \dot{q}_n)(1 + \epsilon \dot{q}_{n+1}) \frac{\mathrm{e}^{q_{n+1} - q_n}}{1 + \epsilon^2 \mathrm{e}^{q_{n+1} - q_n}} - (1 + \epsilon \dot{q}_{n-1})(1 + \epsilon \dot{q}_n) \frac{\mathrm{e}^{q_n - q_{n-1}}}{1 + \epsilon^2 \mathrm{e}^{q_n - q_{n-1}}}$
$\mu = -b = \epsilon^{-1}, \quad \delta = \beta = -\epsilon$	(6)	$\ddot{q}_n = (1 + \epsilon \dot{q}_n) \left(\frac{\dot{q}_{n+1} - \mathrm{e}^{q_{n+1} - q_n}}{1 + \epsilon \mathrm{e}^{q_{n+1} - q_n}} \mathrm{e}^{q_{n+1} - q_n} - \frac{\dot{q}_{n-1} - \mathrm{e}^{q_n - q_{n-1}}}{1 + \epsilon \mathrm{e}^{q_n - q_{n-1}}} \mathrm{e}^{q_n - q_{n-1}} \right)$

Equations (1)–(6) are obtained, respectively, in [13–20]. Though there exist transformations that turn equation (4) into equation (1), equation (5) into equation (2), and equation (6) into equation (3), there are some essential differences among equations (1)–(6). Indeed, equations (4) and (5) reduce to the famous Toda lattice when taking $\epsilon \to 0$. Let $\epsilon \to 0$, equation (6) becomes the lattice discussed in [12]

$$\ddot{q}_n = \dot{q}_{n+1} \mathrm{e}^{q_{n+1} - q_n} - \mathrm{e}^{2(q_{n+1} - q_n)} - \dot{q}_{n-1} \mathrm{e}^{q_n - q_{n-1}} + \mathrm{e}^{2(q_n - q_{n-1})}. \tag{1.7}$$

In this paper, we first give a new integrable lattice soliton system associated with the isospectral problem (1.3) by setting a proper continuous-time evolution equation, then we focus on the integrable discretization of the general Toda-type lattice equation (1.6). As an application, the Lagrangian and Newtonian forms of integrable discretizations of the Toda-type lattice equations (1)–(6) are given uniformly and some new integrable discretizations of lattice equations (1)–(6) are obtained. Finally, the integrable discretization of the integrable lattice system posed in section 2 is discussed.

2. A new integrable lattice soliton system

For the isospectral problem (1.3), we can obtain a novel integrable lattice soliton system by setting the following continuous-time evolution equation:

$$\frac{\mathrm{d}\psi_n}{\mathrm{d}t} = N_n\psi_n \qquad N_n = \frac{1}{1+\lambda^2}\begin{pmatrix} \lambda^2 a_n & \lambda b_n \\ \lambda c_n & \frac{1}{2}(d_n + \lambda^2 e_n) \end{pmatrix} \tag{2.1}$$

where a_n, b_n, c_n, d_n and e_n are determined functions of the potentials $u = (p_n, q_n, r_n, s_n)^T$. From equations (1.3) and (2.1), we obtain

$$N_{n+1}U_n - U_nN_n = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \tag{2.2}$$

where

$$\begin{aligned}
\Delta_{11} &= \lambda p_n Da_n - \frac{\lambda}{1+\lambda^2}((p_n+1)Da_n - r_nb_{n+1} + q_nc_n) \\
\Delta_{12} &= q_na_{n+1} + s_nb_{n+1} - p_nb_n \\
&\quad -\tfrac{1}{2}q_ne_n - \frac{1}{1+\lambda^2}(q_na_{n+1} + s_nb_{n+1} - p_nb_n - b_n + \tfrac{1}{2}q_n(d_n - e_n)) \\
\Delta_{21} &= p_nc_{n+1} - r_na_n - s_nc_n + \tfrac{1}{2}r_ne_{n+1} \\
&\quad + \frac{1}{1+\lambda^2}(r_na_n + s_nc_n - p_nc_{n+1} - c_{n+1} + \tfrac{1}{2}r_n(d_{n+1} - e_{n+1})) \\
\Delta_{22} &= \tfrac{1}{2}\lambda s_nDe_n + \frac{\lambda}{1+\lambda^2}(q_nc_{n+1} - r_nb_n + \tfrac{1}{2}s_n(e_n - d_n) + \tfrac{1}{2}s_n(d_{n+1} - e_{n+1})).
\end{aligned} \tag{2.3}$$

In order to obtain an integrable lattice system, the following equations must be satisfied:

$$\begin{aligned}
&(p_n+1)Da_n - r_nb_{n+1} + q_nc_n = 0 \\
&q_na_{n+1} + s_nb_{n+1} - p_nb_n - b_n + \tfrac{1}{2}q_n(d_n - e_n) = 0 \\
&r_na_n + s_nc_n - p_nc_{n+1} - c_{n+1} + \tfrac{1}{2}r_n(d_{n+1} - e_{n+1}) = 0 \\
&q_nc_{n+1} - r_nb_n + \tfrac{1}{2}s_n(e_n - d_n) + \tfrac{1}{2}s_n(d_{n+1} - e_{n+1}) = 0.
\end{aligned} \tag{2.4}$$

The solvability of equation (2.4) is crucial. Fortunately we can find a solution to equation (2.4) under the assumption $q_nr_n = -(p_n+1)s_n$ and $e_n = -d_n$,

$$\begin{aligned}
a_n &= \frac{p_n+1}{\Delta_n} & b_n &= \frac{q_n}{\Delta_n} \\
c_n &= -\frac{(p_n+1)s_{n-1}}{q_{n-1}\Delta_n} & d_n &= -\frac{q_ns_{n-1}}{q_{n-1}\Delta_n}
\end{aligned} \tag{2.5}$$

where $\Delta_n = 1 + p_n + (q_ns_{n-1}/q_{n-1})$. We thus obtain the following lattice system from equation (1.2):

$$\begin{aligned}
\dot{p}_n &= p_nDa_n \\
\dot{q}_n &= q_na_{n+1} + s_nb_{n+1} - p_nb_n + \tfrac{1}{2}q_nd_n \\
\dot{r}_n &= p_nc_{n+1} - r_na_n - s_nc_n - \tfrac{1}{2}r_nd_{n+1} \\
\dot{s}_n &= -\tfrac{1}{2}s_nDd_n.
\end{aligned}$$

It is easy to show that the consistent condition $\dot{p}_n s_n + (p_n+1)\dot{s}_n = -(\dot{q}_n r_n + q_n \dot{r}_n)$ is satisfied. Therefore, we obtain a new integrable lattice system,

$$\begin{aligned}
\dot{p}_n &= p_n\left(\frac{1+p_{n+1}}{1+p_{n+1}+q_{n+1}s_n/q_n} - \frac{1+p_n}{1+p_n+q_n s_{n-1}/q_{n-1}}\right)\\
\dot{q}_n &= q_n\left(\frac{1+q_n s_{n-1}/2q_{n-1}}{1+p_n+q_n s_{n-1}/q_{n-1}}\right)\\
\dot{s}_n &= \frac{s_n}{2}\left(\frac{q_{n+1}s_n/q_n}{1+p_{n+1}+q_{n+1}s_n/q_n} - \frac{q_n s_{n-1}/q_{n-1}}{1+p_n+q_n s_{n-1}/q_{n-1}}\right).
\end{aligned} \tag{2.6}$$

Under the transformation $p_n \to e^{p_n}, q_n \to e^{q_n}, s_n \to e^{s_n}$, the lattice system (2.6) can be written in the form

$$\begin{aligned}
\dot{p}_n &= \frac{1+e^{p_{n+1}}}{1+e^{p_{n+1}}+e^{q_{n+1}-q_n+s_n}} - \frac{1+e^{p_n}}{1+e^{p_n}+e^{q_n-q_{n-1}+s_{n-1}}}\\
\dot{q}_n &= \frac{1+\frac{1}{2}e^{q_n+s_{n-1}-q_{n-1}}}{1+e^{p_n}+e^{q_n-q_{n-1}+s_{n-1}}}\\
\dot{s}_n &= \frac{1}{2}\left(\frac{e^{q_{n+1}-q_n+s_n}}{1+e^{p_{n+1}}+e^{q_{n+1}-q_n+s_n}} - \frac{e^{q_n-q_{n-1}+s_{n-1}}}{1+e^{p_n}+e^{q_n-q_{n-1}+s_{n-1}}}\right).
\end{aligned} \tag{2.7}$$

Example. We consider a reduction $s_n = \beta =$ constant, $q_n r_n = -\beta(p_n+1)$. In this case, there exists a solution to equation (2.4),

$$\begin{aligned}
a_n &= \frac{p_n+1}{\Delta} \qquad b_n = \frac{q_n}{\Delta} \qquad c_n = -\frac{\beta(p_n+1)}{q_{n-1}\Delta}\\
d_n &= -\frac{2\beta q_n}{q_{n-1}\Delta} + 2\alpha(t) \qquad e_n = 2\alpha(t)
\end{aligned} \tag{2.8}$$

with $\alpha(t)$ an arbitrary function and $\Delta = 1 + p_n + \beta q_n/q_{n-1}$. It follows from equations (2.3), (2.8) and (1.2) that

$$\begin{aligned}
\dot{p}_n &= p_n D a_n\\
\dot{q}_n &= q_n a_{n+1} + \beta b_{n+1} - p_n b_n - \alpha(t) q_n\\
\dot{r}_n &= p_n c_{n+1} - r_n a_n - \beta c_n + \alpha(t) r_n.
\end{aligned}$$

It can be shown that the consistent condition $-\beta\dot{p}_n = \dot{q}_n r_n + q_n \dot{r}_n$ holds identically. We thus obtain the lattice equation

$$\begin{aligned}
\dot{p}_n &= p_n\left(\frac{1+p_{n+1}}{1+p_{n+1}+\beta q_{n+1}/q_n} - \frac{1+p_n}{1+p_n+\beta q_n/q_{n-1}}\right)\\
\dot{q}_n &= q_n\left(\frac{1+\beta q_n/q_{n-1}}{1+p_n+\beta q_n/q_{n-1}} - \alpha(t)\right).
\end{aligned} \tag{2.9}$$

Under the transformation $p_n \to e^{p_n}, q_n \to e^{q_n - \int^t \alpha(t)\,dt}$, equation (2.9) is reduced to the form

$$\begin{aligned}
\dot{p}_n &= \frac{-\beta e^{q_{n+1}-q_n}}{1+e^{p_{n+1}}+\beta e^{q_{n+1}-q_n}} + \frac{\beta e^{q_n-q_{n-1}}}{1+e^{p_n}+\beta e^{q_n-q_{n-1}}}\\
\dot{q}_n &= \frac{1+\beta e^{q_n-q_{n-1}}}{1+e^{p_n}+\beta e^{q_n-q_{n-1}}}
\end{aligned} \tag{2.10}$$

i.e.

$$\ddot{q}_n = \frac{\beta^2 e^{2q_n-2q_{n-1}}}{(1+\beta e^{q_n-q_{n-1}})^2}\dot{q}_n^2(\dot{q}_{n-1}-\dot{q}_n) + (\dot{q}_n-1)\left(\frac{\beta e^{q_n-q_{n-1}}}{1+\beta e^{q_n-q_{n-1}}}\dot{q}_n^2 - \frac{\beta e^{q_{n+1}-q_n}}{1+\beta e^{q_{n+1}-q_n}}\dot{q}_n\dot{q}_{n+1}\right).$$

Equation (2.10) possesses the Hamiltonian structure

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} \qquad \dot{q}_n = \frac{\partial H}{\partial p_n} \tag{2.11}$$

where the Hamiltonian function $H = \sum_n p_n - \sum_n \log(1 + e^{p_n} + \beta e^{q_n - q_{n-1}})$. Two lattice equations (10.11) and (11.14) obtained by Suris in [20] are equivalent to the lattice equation (2.10) essentially. So, they are only special reductions of equation (2.7).

3. Integrable discretizations of the general Toda-type lattice equation (1.6)

In this section, we establish the integrable discretizations of the general Toda-type lattice equation (1.5) or (1.6). As an application, the Lagrangian and Newtonian forms of integrable discretizations of lattice equations (1)–(6) are given uniformly and some new integrable discretizations of lattice equations (1)–(6) are obtained. Given an integrable lattice soliton equation, one would like to construct its integrable discretization. Some examples show that the Lax matrix of the discrete-time approximation coincides with the Lax matrix of the continuous-time system [2, 21–24]. In the difference equations below, we suppose $p_n = p_n(t)$ is a function of the discrete time $t \in h\mathbb{Z}$, and $\tilde{p}_n = p_n(t+h)$, $\underset{\sim}{p}_n = p_n(t-h)$. From Taha and Ablowitz's idea [21], given a proper discrete spectral problem and its discrete-time evolution problem

$$E\psi_n = U_n\psi_n \qquad \tilde{\psi}_n = V_n\psi_n \tag{3.1}$$

the compatibility of equation (3.1) implies the following discrete zero-curvature equation:

$$\tilde{U}_n V_n = V_{n+1} U_n \tag{3.2}$$

with the same matrix U_n as the underlying continuous time spectral problem. If a difference equation derived from equation (3.2) by the proper choice of V_n is a discrete-time approximation of the original continuous-time equation, then the difference equation is called the integrable discretization of the original continuous-time equation. How do we choose a proper V_n? Note that

$$\frac{\tilde{\psi}_n - \psi_n}{h} = \frac{(V_n - I)\psi_n}{h}$$

where I is the unit matrix, we obtain

$$\lim_{h\to 0} \frac{V_n - I}{h} = N_n. \tag{3.3}$$

It is obvious that equation (3.3) is only a necessary condition in order to obtain integrable discretization. Now let us consider problem (3.1) with

$$U_n = \begin{pmatrix} \lambda p_n - \lambda^{-1} & q_n \\ r_n & \lambda\delta \end{pmatrix} \qquad V_n = \begin{pmatrix} a_n + \alpha\lambda^{-2} & \lambda^{-1} b_n \\ \lambda^{-1} c_n & d_n \end{pmatrix} \tag{3.4}$$

where $q_n r_n = \beta p_n - \delta$, β, δ and α are arbitrary constants, a_n, b_n, c_n, d_n are determined. It follows from equation (3.2) that

$$\tilde{U}_n V_n - V_{n+1} U_n = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\Delta_{11} = \lambda(a_n\tilde{p}_n - a_{n+1}p_n) + \lambda^{-1}(Da_n + \alpha(\tilde{p}_n - p_n) + c_n\tilde{q}_n - b_{n+1}r_n)$$
$$\Delta_{12} = b_n\tilde{p}_n + d_n\tilde{q}_n - a_{n+1}q_n - \delta b_{n+1} - \lambda^{-2}(b_n + \alpha q_n)$$
$$\Delta_{21} = \lambda^{-2}(\alpha\tilde{r}_n + c_{n+1}) + \tilde{r}_n a_n + \delta c_n - p_n c_{n+1} - r_n d_{n+1}$$
$$\Delta_{22} = -\lambda\delta D d_n + \lambda^{-1}(b_n\tilde{r}_n - q_n c_{n+1}).$$

We thus obtain that

$$d_n = 1 \qquad b_n = -\alpha q_n \qquad c_n = -\alpha\tilde{r}_{n-1} \tag{3.5}$$

and the following equations:

$$a_n\tilde{p}_n = a_{n+1}p_n \tag{3.6}$$
$$Da_n + \alpha(\tilde{p}_n - p_n) + c_n\tilde{q}_n - b_{n+1}r_n = 0 \tag{3.7}$$
$$b_n\tilde{p}_n + \tilde{q}_n - a_{n+1}q_n - \delta b_{n+1} = 0 \tag{3.8}$$
$$\tilde{r}_n a_n + \delta c_n - p_n c_{n+1} - r_n = 0. \tag{3.9}$$

It follows from equations (3.5)–(3.8) that

$$a_{n+1} = \delta\alpha\frac{q_{n+1}}{q_n} + \frac{\tilde{q}_n}{q_n} - \alpha\tilde{p}_n \qquad c_n = -\frac{Da_n}{\tilde{q}_n} + \alpha\frac{p_n - \tilde{p}_n}{\tilde{q}_n} - \alpha\frac{q_{n+1}r_n}{\tilde{q}_n}. \tag{3.10}$$

Substituting equations (3.5), (3.6) and (3.10) into equation (3.9) and noting $q_n r_n = \beta p_n - \delta$, we find that equation (3.9) is satisfied identically. Therefore, if we choose proper a_n, such that the map derived from equations (3.8) and (3.6) is the discrete-time approximation of equation (1.5), and equation (3.7) holds identically, then the map is an integrable discretization of lattice (1.5). From equation (3.3), we set $\alpha = \mu h$ and

$$a_{n+1} = 1 + bh - \mu h(\beta\tilde{p}_n - \delta)\frac{q_{n+1}}{\tilde{q}_n} + o_{n+1}(h) \tag{3.11}$$

where $o_{n+1}(h)/h \to 0$, as $h \to 0$. It follows from equations (3.8), (3.6) and (3.11) that

$$\frac{\tilde{q}_n - q_n}{h} = \mu q_n\tilde{p}_n - \mu\delta q_{n+1} + q_n\left(b - \mu(\beta\tilde{p}_n - \delta)\frac{q_{n+1}}{\tilde{q}_n} + \frac{o_{n+1}(h)}{h}\right) \tag{3.12}$$

$$\frac{\tilde{p}_n - p_n}{h} = \mu(\beta\tilde{p}_{n-1} - \delta)\frac{q_n\tilde{p}_n}{\tilde{q}_{n-1}} - \mu(\beta\tilde{p}_n - \delta)\frac{q_{n+1}p_n}{\tilde{q}_n} + p_n\left(b + \frac{o_{n+1}(h)}{h}\right) - \tilde{p}_n\left(b + \frac{o_n(h)}{h}\right). \tag{3.13}$$

The map (3.12), (3.13) is a discrete-time approximation of the lattice (1.5). Here, the introduction of the modified term $o_{n+1}(h)$ is important in order to derive an integrable discretization of the lattice (1.5). Then, how do we choose $o_{n+1}(h)$? After some analysis, we let

$$o_{n+1}(h) = -b\mu h^2(\beta\tilde{p}_n - \delta)\frac{q_{n+1}}{\tilde{q}_n} \tag{3.14}$$

and write the map (3.12), (3.13) in the following form:

$$\begin{aligned} h\tilde{p}_n &= \frac{(\tilde{q}_n/q_n - bh - 1)(1 + \delta\mu h(q_{n+1}/\tilde{q}_n))}{\mu(1 - (bh + 1)\beta(q_{n+1}/\tilde{q}_n))} \\ hp_n &= \frac{(\tilde{q}_n/q_n - bh - 1)(1 + \delta\mu h(q_n/\tilde{q}_{n-1}))}{\mu(1 - (bh + 1)\beta(q_n/\tilde{q}_{n-1}))}\,\frac{1 - \beta(q_n/q_{n-1})}{1 - \beta(q_{n+1}/q_n)}. \end{aligned} \tag{3.15}$$

Now we show that equation (3.7), i.e. the following equation,

$$\left(1-(bh+1)\beta\frac{q_{n+1}}{\tilde{q}_n}\right)h\tilde{p}_n-\left(1-\beta\frac{q_{n+1}}{q_n}\right)hp_n-\beta h\tilde{p}_{n-1}\left(\frac{\tilde{q}_n}{\tilde{q}_{n-1}}-(bh+1)\frac{q_n}{\tilde{q}_{n-1}}\right)$$
$$+\delta h\left(\frac{\tilde{q}_n}{\tilde{q}_{n-1}}-\frac{q_{n+1}}{q_n}\right)+\delta h(1+bh)\left(\frac{q_{n+1}}{\tilde{q}_n}-\frac{q_n}{\tilde{q}_{n-1}}\right)=0 \tag{3.16}$$

holds identically. Substituting equation (3.15) into equation (3.16), after some calculations, we obtain

$$\left(1-(bh+1)\beta\frac{q_{n+1}}{\tilde{q}_n}\right)h\tilde{p}_n-\left(1-\beta\frac{q_{n+1}}{q_n}\right)hp_n-\beta h\tilde{p}_{n-1}\left(\frac{\tilde{q}_n}{\tilde{q}_{n-1}}-(bh+1)\frac{q_n}{\tilde{q}_{n-1}}\right)$$
$$=\delta h\left(\frac{q_{n+1}}{q_n}-\frac{\tilde{q}_n}{\tilde{q}_{n-1}}\right)-\delta h(1+bh)\left(\frac{q_{n+1}}{\tilde{q}_n}-\frac{q_n}{\tilde{q}_{n-1}}\right).$$

Equation (3.7) thus holds identically. So, the map (3.15) is an integrable discretization of lattice (1.5). Under the transformation $p_n \to e^{\epsilon p_n}, q_n \to e^{q_n}$, the map (3.15) possesses the beautiful Lagrangian and Newtonian forms, respectively,

$$\begin{aligned} h\,e^{\epsilon\tilde{p}_n} &= \frac{(e^{\tilde{q}_n-q_n}-bh-1)(1+\delta\mu h\,e^{q_{n+1}-\tilde{q}_n})}{\mu(1-(bh+1)\beta e^{q_{n+1}-\tilde{q}_n})} \\ h\,e^{\epsilon p_n} &= \frac{(e^{\tilde{q}_n-q_n}-bh-1)(1+\delta\mu h\,e^{q_n-\tilde{q}_{n-1}})}{\mu(1-(bh+1)\beta e^{q_n-\tilde{q}_{n-1}})}\,\frac{1-\beta e^{q_n-q_{n-1}}}{1-\beta e^{q_{n+1}-q_n}} \end{aligned} \tag{3.17}$$

and

$$\frac{(e^{q_n-\underset{\sim}{q}_n}-bh-1)(1+\delta\mu h\,e^{\underset{\sim}{q}_{n+1}-q_n})}{1-(bh+1)\beta e^{\underset{\sim}{q}_{n+1}-q_n}}=\frac{(e^{\tilde{q}_n-q_n}-bh-1)(1+\delta\mu h\,e^{q_n-\tilde{q}_{n-1}})}{1-(bh+1)\beta e^{q_n-\tilde{q}_{n-1}}}\,\frac{1-\beta e^{q_n-q_{n-1}}}{1-\beta e^{q_{n+1}-q_n}}. \tag{3.18}$$

Example. From the integrable discretization (3.17), (3.18) of the general Toda-type lattice (1.6), integrable discretizations of the lattice equations (1)–(6) are given uniformly as follows in the Lagrangian and Newtonian forms:

$$\begin{aligned} h\,e^{\epsilon\tilde{p}_n} &= (e^{\tilde{q}_n-q_n}-1)(1+h\,e^{q_{n+1}-\tilde{q}_n}) \\ h\,e^{\epsilon p_n} &= (e^{\tilde{q}_n-q_n}-1)(1+h\,e^{q_n-\tilde{q}_{n-1}}) \end{aligned} \tag{3.19}$$

$$\frac{e^{q_n-\underset{\sim}{q}_n}-1}{e^{\tilde{q}_n-q_n}-1}=\frac{1+h\,e^{q_n-\tilde{q}_{n-1}}}{1+h\,e^{\underset{\sim}{q}_{n+1}-q_n}} \tag{3.20}$$

which coincide with integrable discretization for the modified Toda lattice equation (1) obtained in [14].

$$\begin{aligned} h\,e^{\tilde{p}_n} &= \frac{e^{\tilde{q}_n-q_n}-1}{1+g^2e^{q_{n+1}-\tilde{q}_n}} \\ h\,e^{p_n} &= \frac{e^{\tilde{q}_n-q_n}-1}{1+g^2e^{q_n-\tilde{q}_{n-1}}}\,\frac{1+g^2e^{q_n-q_{n-1}}}{1+g^2e^{q_{n+1}-q_n}} \end{aligned} \tag{3.21}$$

$$\frac{e^{q_n-\underset{\sim}{q}_n}-1}{e^{\tilde{q}_n-q_n}-1}=\frac{1+g^2e^{\underset{\sim}{q}_{n+1}-q_n}}{1+g^2e^{q_n-\tilde{q}_{n-1}}}\,\frac{1+g^2e^{q_n-q_{n-1}}}{1+g^2e^{q_{n+1}-q_n}} \tag{3.22}$$

which coincide with integrable discretization for the relativistic Toda lattice (2) obtained in [24].

$$
\begin{aligned}
h\,\mathrm{e}^{\tilde{p}_n} &= \frac{\mathrm{e}^{\tilde{q}_n-q_n}-1}{1+g^2\mathrm{e}^{q_{n+1}-\tilde{q}_n}}(1+\delta g^2 h\,\mathrm{e}^{q_{n+1}-\tilde{q}_n}) \\
h\,\mathrm{e}^{p_n} &= \frac{\mathrm{e}^{\tilde{q}_n-q_n}-1}{1+g^2\mathrm{e}^{q_n-\tilde{q}_{n-1}}}\,\frac{1+g^2\mathrm{e}^{q_n-q_{n-1}}}{1+g^2\mathrm{e}^{q_{n+1}-q_n}}(1+\delta g^2 h\,\mathrm{e}^{q_n-\tilde{q}_{n-1}})
\end{aligned} \tag{3.23}
$$

$$
\frac{\mathrm{e}^{q_n-\underset{\sim}{q}_n}-1}{\mathrm{e}^{\tilde{q}_n-q_n}-1} = \frac{1+g^2\mathrm{e}^{\underset{\sim}{q}_{n+1}-q_n}}{1+g^2\mathrm{e}^{q_n-\tilde{q}_{n-1}}}\,\frac{1+g^2\mathrm{e}^{q_n-q_{n-1}}}{1+g^2\mathrm{e}^{q_{n+1}-q_n}}\,\frac{1+\delta g^2 h\,\mathrm{e}^{q_n-\tilde{q}_{n-1}}}{1+\delta g^2 h\,\mathrm{e}^{\underset{\sim}{q}_{n+1}-q_n}} \tag{3.24}
$$

which are just results for equation (3) obtained in [20]. Equations (3.23) and (3.24) reduce to equations (3.21) and (3.22) if we let $\delta = 0$.

$$
\begin{aligned}
h\,\mathrm{e}^{\epsilon\tilde{p}_n} &= \epsilon\left(\mathrm{e}^{\tilde{q}_n-q_n}+\frac{h}{\epsilon}-1\right)(1+\epsilon h\,\mathrm{e}^{q_{n+1}-\tilde{q}_n}) \\
h\,\mathrm{e}^{\epsilon p_n} &= \epsilon\left(\mathrm{e}^{\tilde{q}_n-q_n}+\frac{h}{\epsilon}-1\right)(1+\epsilon h\,\mathrm{e}^{q_n-\tilde{q}_{n-1}})
\end{aligned} \tag{3.25}
$$

$$
\frac{\epsilon(\mathrm{e}^{q_n-\underset{\sim}{q}_n}-1)+h}{\epsilon(\mathrm{e}^{\tilde{q}_n-q_n}-1)+h} = \frac{1+\epsilon h\,\mathrm{e}^{q_n-\tilde{q}_{n-1}}}{1+\epsilon h\,\mathrm{e}^{\underset{\sim}{q}_{n+1}-q_n}} \tag{3.26}
$$

which coincide with the results for equation (4) obtained in [13]. If $\epsilon = h$, equations (3.25) and (3.26) reduce to

$$
\begin{aligned}
\mathrm{e}^{h\tilde{p}_n} &= \mathrm{e}^{\tilde{q}_n-q_n}(1+h^2\mathrm{e}^{q_{n+1}-\tilde{q}_n}) \\
\mathrm{e}^{hp_n} &= \mathrm{e}^{\tilde{q}_n-q_n}(1+h^2\mathrm{e}^{q_n-\tilde{q}_{n-1}})
\end{aligned} \tag{3.27}
$$

$$
\mathrm{e}^{\tilde{q}_n-2q_n+\underset{\sim}{q}_n} = \frac{1+h^2\mathrm{e}^{\underset{\sim}{q}_{n+1}-q_n}}{1+h^2\mathrm{e}^{q_n-\tilde{q}_{n-1}}} \tag{3.28}
$$

which are just the Lagrangian and Newtonian forms of the integrable discretization of the Toda lattice.

$$
\begin{aligned}
h\,\mathrm{e}^{\epsilon\tilde{p}_n} &= \frac{\epsilon\mathrm{e}^{\tilde{q}_n-q_n}+h-\epsilon}{1+(\epsilon^2-\epsilon h)\mathrm{e}^{q_{n+1}-\tilde{q}_n}} \\
h\,\mathrm{e}^{\epsilon p_n} &= \frac{\epsilon\mathrm{e}^{\tilde{q}_n-q_n}+h-\epsilon}{1+(\epsilon^2-\epsilon h)\mathrm{e}^{q_n-\tilde{q}_{n-1}}}\,\frac{1+\epsilon^2\mathrm{e}^{q_n-q_{n-1}}}{1+\epsilon^2\mathrm{e}^{q_{n+1}-q_n}}
\end{aligned} \tag{3.29}
$$

$$
\frac{\epsilon\mathrm{e}^{q_n-\underset{\sim}{q}_n}+h-\epsilon}{\epsilon\mathrm{e}^{\tilde{q}_n-q_n}+h-\epsilon} = \frac{1+(\epsilon^2-\epsilon h)\mathrm{e}^{\underset{\sim}{q}_{n+1}-q_n}}{1+(\epsilon^2-\epsilon h)\mathrm{e}^{q_n-\tilde{q}_{n-1}}}\,\frac{1+\epsilon^2\mathrm{e}^{q_n-q_{n-1}}}{1+\epsilon^2\mathrm{e}^{q_{n+1}-q_n}} \tag{3.30}
$$

which are integrable discretizations for equation (5). If $\epsilon = h$, equations (3.29) and (3.30) reduce to

$$
\begin{aligned}
\mathrm{e}^{h\tilde{p}_n} &= \mathrm{e}^{\tilde{q}_n-q_n} \\
\mathrm{e}^{hp_n} &= \frac{\mathrm{e}^{\tilde{q}_n-q_n}+h^2\mathrm{e}^{\tilde{q}_n-q_{n-1}}}{1+h^2\mathrm{e}^{q_{n+1}-q_n}}
\end{aligned} \tag{3.31}
$$

$$
\mathrm{e}^{2q_n-\underset{\sim}{q}_n-\tilde{q}_n} = \frac{1+h^2\mathrm{e}^{q_n-q_{n-1}}}{1+h^2\mathrm{e}^{q_{n+1}-q_n}} \tag{3.32}
$$

which are new Lagrangian and Newtonian forms of the integrable discretization of the Toda lattice.

$$
\begin{aligned}
h\,\mathrm{e}^{\epsilon\tilde{p}_n} &= \frac{(\epsilon\mathrm{e}^{\tilde{q}_n-q_n}+h-\epsilon)(1-h\,\mathrm{e}^{q_{n+1}-\tilde{q}_n})}{1+(\epsilon-h)\mathrm{e}^{q_{n+1}-\tilde{q}_n}} \\
h\,\mathrm{e}^{\epsilon p_n} &= \frac{(\epsilon\mathrm{e}^{\tilde{q}_n-q_n}+h-\epsilon)(1-h\,\mathrm{e}^{q_n-\tilde{q}_{n-1}})(1+\epsilon\mathrm{e}^{q_n-q_{n-1}})}{(1+(\epsilon-h)\mathrm{e}^{q_n-\tilde{q}_{n-1}})(1+\epsilon\mathrm{e}^{q_{n+1}-q_n})}
\end{aligned}
\tag{3.33}
$$

$$
\frac{(\epsilon\mathrm{e}^{q_n-\underset{\sim}{q}_n}+h-\epsilon)(1-h\,\mathrm{e}^{\underset{\sim}{q}_{n+1}-q_n})}{1+(\epsilon-h)\mathrm{e}^{\underset{\sim}{q}_{n+1}-q_n}} = \frac{(\epsilon\mathrm{e}^{\tilde{q}_n-q_n}+h-\epsilon)(1-h\,\mathrm{e}^{q_n-\tilde{q}_{n-1}})(1+\epsilon\mathrm{e}^{q_n-q_{n-1}})}{(1+(\epsilon-h)\mathrm{e}^{q_n-\tilde{q}_{n-1}})(1+\epsilon\mathrm{e}^{q_{n+1}-q_n})}
\tag{3.34}
$$

which are integrable discretizations for equation (6). If $\epsilon = h$, equations (3.33) and (3.34) reduce to

$$
\begin{aligned}
\mathrm{e}^{h\tilde{p}_n} &= \mathrm{e}^{\tilde{q}_n-q_n}(1-h\,\mathrm{e}^{q_{n+1}-\tilde{q}_n}) \\
\mathrm{e}^{hp_n} &= \mathrm{e}^{\tilde{q}_n-q_n}(1-h\,\mathrm{e}^{q_n-\tilde{q}_{n-1}})\frac{1+h\mathrm{e}^{q_n-q_{n-1}}}{1+h\mathrm{e}^{q_{n+1}-q_n}}
\end{aligned}
\tag{3.35}
$$

$$
\mathrm{e}^{2q_n-\underset{\sim}{q}_n-\tilde{q}_n} = \frac{(1-h\,\mathrm{e}^{q_n-\tilde{q}_{n-1}})(1+h\mathrm{e}^{q_n-q_{n-1}})}{(1-h\,\mathrm{e}^{\underset{\sim}{q}_{n+1}-q_n})(1+h\mathrm{e}^{q_{n+1}-q_n})}
\tag{3.36}
$$

which coincide with the Lagrangian and Newtonian forms of integrable discretization of lattice equation (1.7) obtained in [13]. A question arises: is a_{n+1} presented by equations (3.11) and (3.14) unique? The answer is negative. Suppose

$$
a_{n+1} = 1 + h(b+\gamma h) - \mu h(1+bh+\gamma h^2)(\beta\tilde{p}_n-\delta)\frac{q_{n+1}}{\tilde{q}_n}
\tag{3.37}
$$

where γ is an arbitrary constant. It is obvious that the map derived from equations (3.8) and (3.6) with equation (3.37) is a discrete-time approximation of the lattice equation (1.5). It follows from equations (3.8), (3.6) and (3.37) that

$$
\begin{aligned}
h\tilde{p}_n &= \frac{(\tilde{q}_n/q_n-1-bh-\gamma h^2)(1+\delta\mu h(q_{n+1}/\tilde{q}_n))}{\mu(1-(1+bh+\gamma h^2)\beta(q_{n+1}/\tilde{q}_n))} \\
hp_n &= \frac{(\tilde{q}_n/q_n-1-bh-\gamma h^2)(1+\delta\mu h(q_n/\tilde{q}_{n-1}))}{\mu(1-(1+bh+\gamma h^2)\beta(q_n/\tilde{q}_{n-1}))}\,\frac{1-\beta(q_n/q_{n-1})}{1-\beta(q_{n+1}/q_n)}
\end{aligned}
\tag{3.38}
$$

which is another discrete-time approximation of lattice (1.5). Now we show that equation (3.7), i.e. the following equation,

$$
\begin{aligned}
&\left(1-(1+bh+\gamma h^2)\beta\frac{q_{n+1}}{\tilde{q}_n}\right)h\tilde{p}_n - \left(1-\beta\frac{q_{n+1}}{q_n}\right)hp_n + \beta h(1+bh+\gamma h^2)\frac{\tilde{p}_{n-1}q_n}{\tilde{q}_{n-1}} \\
&\qquad -\beta h\frac{\tilde{p}_{n-1}\tilde{q}_n}{\tilde{q}_{n-1}} + \delta h\left(\frac{\tilde{q}_n}{\tilde{q}_{n-1}}-\frac{q_{n+1}}{q_n}\right) + \delta h(1+bh+\gamma h^2)\left(\frac{q_{n+1}}{\tilde{q}_n}-\frac{q_n}{\tilde{q}_{n-1}}\right) = 0
\end{aligned}
\tag{3.39}
$$

is satisfied. Substituting equation (3.38) into equation (3.39), after some calculations, we know that equation (3.7) holds identically. So, the map (3.38) is also an integrable discretization of lattice (1.5). Under transformation $p_n \to \mathrm{e}^{\epsilon p_n}, q_n \to \mathrm{e}^{q_n}$, the map (3.38) possesses the beautiful Lagrangian and Newtonian forms, respectively,

$$
\begin{aligned}
h\,\mathrm{e}^{\epsilon\tilde{p}_n} &= \frac{(\mathrm{e}^{\tilde{q}_n-q_n}-1-bh-\gamma h^2)(1+\delta\mu h\,\mathrm{e}^{q_{n+1}-\tilde{q}_n})}{\mu(1-(1+bh+\gamma h^2)\beta\mathrm{e}^{q_{n+1}-\tilde{q}_n})} \\
h\,\mathrm{e}^{\epsilon p_n} &= \frac{(\mathrm{e}^{\tilde{q}_n-q_n}-1-bh-\gamma h^2)(1+\delta\mu h\,\mathrm{e}^{q_n-\tilde{q}_{n-1}})}{\mu(1-(1+bh+\gamma h^2)\beta\mathrm{e}^{q_n-\tilde{q}_{n-1}})}\,\frac{1-\beta\mathrm{e}^{q_n-q_{n-1}}}{1-\beta\mathrm{e}^{q_{n+1}-q_n}}
\end{aligned}
\tag{3.40}
$$

and

$$\frac{(\mathrm{e}^{q_n-\underset{\sim}{q}_n}-1-bh-\gamma h^2)(1+\delta\mu h\,\mathrm{e}^{\underset{\sim}{q}_{n+1}-q_n})}{1-(1+bh+\gamma h^2)\beta\mathrm{e}^{\underset{\sim}{q}_{n+1}-q_n}}$$
$$=\frac{(\mathrm{e}^{\tilde{q}_n-q_n}-1-bh-\gamma h^2)(1+\delta\mu h\,\mathrm{e}^{q_n-\tilde{q}_{n-1}})}{1-(1+bh+\gamma h^2)\beta\mathrm{e}^{q_n-\tilde{q}_{n-1}}}\,\frac{1-\beta\mathrm{e}^{q_n-q_{n-1}}}{1-\beta\mathrm{e}^{q_{n+1}-q_n}}. \tag{3.41}$$

From the map (3.40)–(3.41), new integrable discretizations of lattice equation (1)–(6) are obtained.

4. Conclusion and discussion

We have proposed a novel integrable lattice system associated with the discrete isospectral problem (1.3). Integrable discretizations of the general Toda-type lattice equation (1.5) or (1.6) are established and the Lagrangian and Newtonian forms of integrable discretizations of Toda-type lattices (1)–(6) are derived uniformly and some new integrable discretizations of lattices (1)–(6) are given. We ask the question of how to obtain an integrable discretization of the lattice system (2.6)? Is the method posed in section 3 applicable to lattice system (2.6)? Following the method above, by condition (3.3), we consider problem (3.1) with

$$U_n=\begin{pmatrix}\lambda p_n-\lambda^{-1} & q_n\\ r_n & \lambda s_n\end{pmatrix}$$
$$V_n=\frac{1}{1+\lambda^2}\begin{pmatrix}1+\lambda^2+\lambda^2 f_n & \lambda u_n\\ \lambda v_n & 1+\lambda^2+\frac{1}{2}(w_n+\lambda^2 g_n)\end{pmatrix} \tag{4.1}$$

where $q_n r_n=-(p_n+1)s_n$, functions f_n, g_n, u_n, v_n and w_n are determined. It follows from the discrete zero curvature equation (3.2) that

$$u_n=\tfrac{1}{2}w_n\tilde{q}_n+\tilde{q}_n-q_n \tag{4.2}$$
$$v_{n+1}=\tfrac{1}{2}w_{n+1}r_n+r_n-\tilde{r}_n \tag{4.3}$$
$$(f_n+1)\tilde{p}_n=(f_{n+1}+1)p_n \tag{4.4}$$
$$u_n\tilde{p}_n+(\tfrac{1}{2}g_n+1)\tilde{q}_n=(f_{n+1}+1)q_n+u_{n+1}s_n \tag{4.5}$$
$$(\tfrac{1}{2}g_n+1)\tilde{s}_n=(\tfrac{1}{2}g_{n+1}+1)s_n \tag{4.6}$$
$$Df_n+\tilde{p}_n-p_n+v_n\tilde{q}_n-u_{n+1}r_n=0 \tag{4.7}$$
$$(f_n+1)\tilde{r}_n+v_n\tilde{s}_n-v_{n+1}p_n-(\tfrac{1}{2}g_{n+1}+1)r_n=0 \tag{4.8}$$
$$u_n\tilde{r}_n+(\tfrac{1}{2}w_n+1)\tilde{s}_n-v_{n+1}q_n-(\tfrac{1}{2}w_{n+1}+1)s_n=0. \tag{4.9}$$

From condition (3.3), we let

$$w_n=hd_n+o_{1,n}(h)\qquad g_n=-w_n\qquad \frac{o_{1,n}(h)}{h}\to 0\quad (h\to 0) \tag{4.10}$$

where d_n is presented by equation (2.5). Then,

$$\frac{w_n}{h}\to d_n\qquad \frac{g_n}{h}\to e_n\quad (h\to 0).$$

Note that with the equations for $\dot{q}_n$ and $\dot{r}_n$, we can prove

$$\frac{u_n}{h}\to b_n\qquad \frac{v_n}{h}\to c_n\quad (h\to 0)$$

where b_n and c_n are presented by equation (2.5). Let

$$f_n = ha_n + o_{2,n}(h) \qquad \frac{o_{2,n}(h)}{h} \to 0 \quad (h \to 0). \tag{4.11}$$

From (4.4)–(4.6) it follows that

$$\frac{\tilde{p}_n - p_n}{h} = \left(\frac{1 + p_{n+1}}{\Delta_{n+1}}\right)p_n - \left(\frac{1 + p_n}{\Delta_n}\right)\tilde{p}_n + \frac{o_{2,n+1}(h)p_n - o_{2,n}(h)\tilde{p}_n}{h} \tag{4.12}$$

$$\frac{\tilde{q}_n - q_n}{h} = \left(\frac{1 + p_{n+1}}{\Delta_{n+1}}\right)q_n - \frac{q_n s_{n-1}\tilde{q}_n}{2q_{n-1}\Delta_n} + \frac{u_{n+1}s_n - u_n\tilde{p}_n}{h} + \frac{o_{2,n+1}(h)q_n}{h} + \frac{o_{1,n}(h)\tilde{q}_n}{2h} \tag{4.13}$$

$$\frac{\tilde{s}_n - s_n}{h} = \frac{q_{n+1}s_n^2}{2q_n\Delta_{n+1}} - \frac{q_n s_{n-1}\tilde{s}_n}{2q_{n-1}\Delta_n} - \frac{o_{1,n+1}(h)s_n - o_{1,n}(h)\tilde{s}_n}{2h}. \tag{4.14}$$

The map (4.12)–(4.14) is a discrete-time approximation of the lattice system (2.6). In order to obtain the integrable discretization of the lattice system (2.6), we must choose proper modified terms $o_{1,n}(h)$ and $o_{2,n}(h)$ such that equations (4.7)–(4.9) hold identically. However, we failed in finding proper $o_{1,n}(h)$ and $o_{2,n}(h)$. Recently, by the singularity confinement method [25–27], the bilinear form of the discrete-time relativistic Toda lattice equations was established and the N-soliton solution was constructed explicitly by Maruno *et al* in the form of the Casorati determinant [28]. So we believe that the search for the bilinear forms and N-soliton solutions for the discrete-time general Toda-type lattice and integrable discretization of the lattice system (2.6) are worth further future effort.

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